Introduction to Gauge theory

No lectures on 17/5, 14/6

replacement ones : 30/5 , 20/6

$\boxed{10:00 - 12:00}$

Goal : Uhlenbeck compactification

Donaldson invariants

Plan:

- connections, curvatures
- Yang-Mills connections, Anti-Self-dual instantons
- Sobolev spaces, Elliptic operators
- Atiyah-Hitchin-Singer complex, Kuranishi model
- Freed-Uhlenbeck perturbation, orientation
- Uhlenbeck compactness, Taubes gluing
- Uhlenbeck compactification, Gieseker compactification
- Donaldson invariants.

§1 Connections and Curvatures.

§1.1. Connections on a vector bundle.

<u>Def</u>

Let $E \to X$: vector bundle over a smooth manifold X.

A linear differential operator

$$\nabla : \Gamma(E) \longrightarrow \Gamma(E \otimes T^*X)$$

is a <u>covariant derivative</u> or <u>connection</u>.

if it satisfies

$$\nabla(fs) = df \otimes s + f\nabla s, \quad \text{(Leibniz rule)}$$

where $f \in \Omega^0(X)$, $s \in \Gamma(E)$,

(<u>Rmk</u>: $\nabla s(v) = \nabla_v s$ for $v \in T_x X$

i.e. $\nabla_v : \Gamma(E) \to E_x$

is said to be the <u>covariant derivative</u> <u>in the direction of v at x</u>.)

Local description

Take $U \subset X$, an open set.

$e_1, \cdots, e_r$: a local frame of E on U

$r = \operatorname{rank} E$.

Write a section s of E locally.

$$s = \sum_{i=1}^{r} s^i e_i \quad , \quad s^i \in \Omega^0(U)$$

As $\nabla e_i \in \Gamma(E \otimes T^*X)$,

one can write

$$\nabla e_i = \sum_{j=1}^{r} A_i^{\,j} e_j ,$$

where A^i_j is a locally defined 1-form.

Then ∇s is written locally as

$$\nabla s = \sum \left(ds^i + \sum A^i_j s^j \right) e_i$$

we call $A = (A^i_j)$ the <u>connection 1-form</u> of ∇

<u>Prop.</u>

Let $\{U_\alpha\}$: open cover of X

$\{\psi_{\alpha\beta}\}$: transition function of E

Then

$$A_\beta = \psi_{\alpha\beta}^{-1} A_\alpha \psi_{\alpha\beta} + \psi_{\alpha\beta}^{-1} d\psi_{\alpha\beta} \quad (*)$$

on $U_\alpha \cap U_\beta$,

where $A_\alpha := A|_{U_\alpha}$, $A_\beta := A|_{U_\beta}$.

<u>Prop.</u>

To give a conn on E is the same as

to give a matrix-valued 1-form A with the above condition $(*)$

<u>Excersise</u> : prove these.

We sometimes denote a connection by A.

Covariant exterior differentiation

For $s\xi$, where $s \in \Gamma(E)$, $\xi \in \Omega^p(X)$

define $d_A(s\xi)$ by

$$d_A(s\xi) = \nabla s \wedge \xi + s\, d\xi$$

This defines an operator

$$d_A : \Omega^p(E) \to \Omega^{p+1}(E)$$

We call this <u>covariant exterior derivative</u>

<u>Property</u>

$$d_A(\varphi \wedge \psi) = (d_A\varphi)\wedge\psi - (-1)^p \varphi \wedge d\psi \quad \cdots (**)$$

for $\varphi \in \Omega^p(E)$, $\psi \in \Omega^q(X)$

<u>Question</u>

$$d_A^2 = ? \qquad (\text{cf. } d^2 = 0)$$

For $s \in \Gamma(E)$, $\xi \in \Omega^p(X)$, we get.

$$\begin{aligned} d_A^2(s\xi) &= d_A^2 s \wedge \xi - d_A s \wedge d\xi + d_A s \wedge d\xi \\ &\quad + s\, d^2\xi \\ &= d_A^2 s \wedge \xi \end{aligned}$$

So enough to look at d_A^2 on $\Gamma(E)$

Lem

d_A^2 is an algebraic oprator.

i.e. $d_A^2 \in \Omega^2(\mathrm{End}(E))$, $\mathrm{End}(E) = E^* \otimes E$.

$\because$) For $f \in \Omega^0(X)$, $s \in \Omega^0(E)$,

we have

$$d_A^2(fs) = f \wedge d_A^2 s$$

(put $\xi = s$. in (**))

Thus $(d_A^2 s)_x$ is determined only by s_x $\square$.

Def

We write $F_A := d_A^2$ and call it

the curvature of A

Properties of curvature

(1) $F_A(X, Y)s = \frac{1}{2}(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})s$

$s \in \Gamma(E)$, $X, Y \subset TX$.

(Ricci's identity)

(2) $d_A F_A = 0$ (Bianchi identity)

(3) Locally

$$d_A^2 e_i = \sum(dA^i_j - \sum A^k_i \wedge A^j_k) e_j$$

§1.2 Connections on a principal bundle.

want to think about more general structure group G.

Let G : Lie group.

$P \to X$, a principal G-bundle over a smooth manifold X.

<u>Def</u>. Denote by $\mathfrak{g}$ Lie algebra of G.

A $\mathfrak{g}$-valued 1-form A on P is said to be a <u>connection</u> on P if

it is right equivariant and the identity on vertical directions, i.e., it satisfies

$$\begin{cases} \cdot\ R_g^* A = g^{-1} A g, & g \in G \\ \cdot\ A(\xi^*) = \xi, & \xi \in \mathfrak{g}, \end{cases}$$

where $\cdot\ R_g : P \to P$, $p \mapsto pg$, $p \in P$

$\cdot\ \xi^*$ is a vector field on P generated by ξ (by $e^{t\xi}$)

Rmk one can define a connection on P by giving an equivariant horizontal distribution $H \subset TP$.

The correspondence with the above definition is given by $H_p = \mathrm{Ker}\, A_p$ for each $p \in P$.

Def. We define the curvature of a connection A by

$$F_A := dA + \frac{1}{2}[A \wedge A]$$

(as a $\mathfrak{g}$-valued 2-form.)

Excercises

(1) Given a reference connection, a connection on P can be identified with an $ad(P)$-valued ($:= P \times_{Ad} \mathfrak{g}$) 1-form on X

(2) F_A can be identified with $ad(P)$-valued 2-form on X.

(3) Let $\begin{cases} P \to X, \text{ principal } G\text{-bundle with structure group } G \\ \rho : G \to GL(r, \mathbb{V}), \text{ a representation of } G. \quad \mathbb{V} = \mathbb{R}, \mathbb{C} \end{cases}$

consider the associated bundle $E := P \times_\rho \mathbb{V}^r$.

Then a connection on P induces a covariant derivative $\mathbb{D} = \mathbb{D}^A$ on E

(this is similar to (1) and (2))

§1.3 Gauge group and its action on connections

Def. An equivariant map $u : P \to G$, i.e.

$$u(pg) = g^{-1} u(p) g, \quad p \in P, \ g \in G$$

is called a gauge transformation.

(11) This is identified with a fibre-preserving bundle automorphism $\varphi : P \to P$ with $\varphi(pg) = \varphi(p) g$ for $p \in P$, $g \in G$.

$\because$) for a given u, define

$\varphi : P \to P$ by $p\,u(p)$ for each $p \in P$.

Then φ satisfies

$$\varphi(pg) = pg \cdot u(pg) = pg \cdot g^{-1} u(p)\, g$$

$$= p\, u(p)\, g = \varphi(p)\, g \qquad \square$$

- We denote by $\mathcal{G}_P$ all gauge transformations and call it <u>group of gauge transformations</u> or <u>gauge group</u>.

This is an infinite dimensional Lie group.

<u>Excercise</u>

(1) $\mathcal{G}_P$ is identified with $\Gamma(P \times_{Ad} G)$

(2) The Lie algebra of $\mathcal{G}_P$ is given by $\Gamma(P \times_{Ad} \mathfrak{g})$

Gauge group action

Consider $A \mapsto \varphi(A) := \varphi^* A$

for $\varphi \in \mathcal{G}_P$, $A \in \mathcal{A}_P$

Lem

$\varphi(A) \in \mathcal{A}_P$.

(Hint: $\varphi \circ R_g = R_g \circ \varphi$, $g \in G$)

Then this gives an action $\mathcal{G}_P \curvearrowright \mathcal{A}_P$

Locally on an open subset $U \subset X$,
the gauge group action is written as

$$\varphi_U^{-1} A_U \varphi_U + \varphi_U^{-1} d\varphi_U ,$$

where $A_U := A|_U$, $A \in \mathcal{A}_P$, $\varphi_U := \varphi|_U$,
$\varphi \in \mathcal{G}_P$.

Excercise

$$F_{\varphi(A)} = \varphi^{-1} F_A \varphi$$

Configuration space

$$\mathcal{B}_P := \mathcal{A}_P / \mathcal{G}_P$$

we are interested in "Geometry" and "Topology" of $\mathcal{B}_P$

Next week:

Yang-Mills functional : $\mathcal{B}_P \to \mathbb{R}$.

Anti-self-dual instantons